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Vector Representation of Interacting Dirac Equation

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Using the Clifford algebra, a vectorial equation for the Dirac spinorial equation is constructed and the relation with the Klein–Gordon equation becomes transparent. The equation interacting with the electromagnetic field leads to a nontrivial generalization for the interacting Klein–Gordon equation. The Lagrangian density for this interaction is given.

1. INTRODUCTION

The Dirac equation is a cornerstone of quantum theory (Schweber, 1964; Bogoliubov and Shirkov, 1959). The purpose of the present paper is to find a vectorial representation of the interacting Dirac equation. The purpose is to attain a full freedom in choosing the traditional spinorial representation or to shift to a vectorial notation with the same physical content. The interacting case produces a nontrivial generalization of the Klein–Gordon equation and an interacting Lagrangian density that is different than the one used in the literature.

2. THE DIRAC MATRICES

In the sequel, I is the 4×4 unit matrix, $g^{\mu\nu}$ is the Minkowski metric tensor, and the indexes μ and ν run from 0 to 3. The Dirac 4×4 matrices are defined by means of the relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I$. In terms of the Levi-Civita antisymmetric tensor, we define new matrices as follows:

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$$\epsilon_{\alpha\kappa\mu\nu} = \begin{cases} 1 & \text{if} \quad \alpha, \kappa, \mu, \nu \text{ is an even permutation of } 0\ 1\ 2\ 3 \\ -1 & \text{if} \quad \alpha, \kappa, \mu, \nu \text{ is an odd permutation of } 0\ 1\ 2\ 3 \\ 0 & \text{in other cases} \end{cases}$$

$$\sigma_{\alpha\kappa} = -\sigma_{\kappa\alpha} = \tfrac{1}{2}\, \epsilon_{\alpha\kappa\mu\nu}\, \gamma^{\mu}\gamma^{\nu}$$

$$\sigma_{\alpha} = \tfrac{1}{3!}\, \epsilon_{\alpha\kappa\mu\nu}\gamma^{\kappa}\gamma^{\mu}\gamma^{\nu}, \qquad \sigma = \tfrac{1}{4!}\, \epsilon_{\alpha\kappa\mu\nu}\gamma^{\alpha}\gamma^{\kappa}\gamma^{\mu}\gamma^{\nu}$$

Together with the unit matrix I and with the γ^{μ} matrices, they form a set of 16 matrices. Any 4×4 square matrix Φ can be written as a linear combination of these 16 matrices

$$\Phi = BI + B_{\alpha}\gamma^{\alpha} + B_{\alpha\kappa}\sigma^{\alpha\kappa} + \tilde{B}^{\alpha}\sigma_{\alpha} + \tilde{B}\sigma \tag{1}$$

The Clifford algebra product of any two of the 16 matrices is given as follows:

$$\gamma^{\alpha}\gamma^{\kappa} = g^{\alpha\kappa}I - \tfrac{1}{2}\,\epsilon^{\alpha\kappa\mu\nu}\sigma_{\mu\nu}, \qquad \sigma_{\alpha}\sigma_{\kappa} = g_{\alpha\kappa}I - \tfrac{1}{2}\,\epsilon_{\alpha\kappa\mu\nu}\sigma^{\mu\nu}$$

$$\gamma^{\alpha}\sigma_{\kappa} = \sigma\delta^{\alpha}_{\kappa} - g^{\alpha\mu}\sigma_{\mu\kappa}, \qquad \sigma_{\kappa}\gamma^{\alpha} = -\sigma\delta^{\alpha}_{\kappa} - g^{\alpha\mu}\sigma_{\mu\kappa}$$

$$\gamma^{\lambda}\sigma_{\mu\nu} = \sigma_{\mu}\delta^{\lambda}_{\nu} - \sigma_{\nu}\delta^{\lambda}_{\mu} + \epsilon_{\mu\nu\alpha\kappa}g^{\lambda\alpha}\gamma^{\kappa}$$

$$\sigma_{\mu\nu}\gamma^{\lambda} = \sigma_{\mu}\delta^{\lambda}_{\nu} - \sigma_{\nu}\delta^{\lambda}_{\mu} - \epsilon_{\mu\nu\alpha\kappa}g^{\lambda\alpha}\gamma^{\kappa}$$

$$\sigma_{\lambda}\sigma_{\mu\nu} = \gamma_{\nu}g_{\mu\lambda} - \gamma_{\mu}g_{\nu\lambda} + \epsilon_{\mu\nu\lambda\kappa}\sigma^{\kappa}$$

$$\sigma_{\mu\nu}\sigma_{\lambda} = \gamma_{\nu}g_{\mu\lambda} - \gamma_{\mu}g_{\nu\lambda} - \epsilon_{\mu\nu\lambda\kappa}\sigma^{\kappa}$$

$$\sigma\sigma_{\mu\nu} = -\tfrac{1}{2}\,\epsilon_{\mu\nu\alpha\kappa}\sigma^{\alpha\kappa}, \qquad \sigma_{\mu\nu}\sigma = \sigma\sigma_{\mu\nu} = -\tfrac{1}{2}\,\epsilon_{\mu\nu\alpha\kappa}\sigma^{\alpha\kappa}$$

$$\sigma_{\mu\nu}\sigma_{\alpha\kappa} = (g_{\mu\alpha}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\alpha})I + \epsilon_{\alpha\kappa\mu\nu}\sigma - \epsilon_{\alpha\kappa\nu\lambda}\sigma^{\lambda}{}_{\mu} + \epsilon_{\alpha\kappa\mu\lambda}\sigma^{\lambda}{}_{\nu}$$

$$\gamma^{\mu}\sigma = \sigma^{\mu}, \qquad \sigma\gamma^{\mu} = -\sigma^{\mu}, \qquad \sigma^{\mu}\sigma = -\gamma^{\mu}, \qquad \sigma\sigma^{\mu} = \gamma^{\mu}, \qquad \sigma\sigma = -I$$

The 16 matrices together with their negatives form a group of 32 elements. Therefore the above equations form the multiplication table of the group and although these products were written with several terms, only one of them is different from zero.

3. LORENTZ TRANSFORMATIONS

The dual to the antisymmetric tensor $F_{\alpha\nu}$ in Minkowski space is $\check{F}_{\alpha\kappa} = \frac{1}{2}\, \epsilon_{\alpha\kappa\mu\nu}\, F^{\mu\nu}$, and the two invariants of the $F_{\mu\nu}$ are (Synge, 1956; Piña, 1967)

$$F^{\alpha\kappa}F_{\alpha\kappa} = 2H\cos\theta, \qquad \check{F}^{\alpha\kappa}F_{\alpha\kappa} = 2H\sin\theta$$

The *null case* is when both invariants are zero: $H = 0$. Then $F^{\alpha}{}_{\kappa}\, F^{\kappa}{}_{\mu}\, F^{\mu}{}_{\nu} = 0$. In the nonnull case we define

$$\omega_{\alpha\kappa} = \frac{1}{\sqrt{H}}\left(\sin\frac{\theta}{2} F_{\alpha\kappa} - \cos\frac{\theta}{2} \check{F}_{\alpha\kappa}\right)$$

$$\omega^{\alpha}{}_{\kappa}\, \omega^{\kappa}{}_{\mu}\, \omega^{\mu}{}_{\nu} = \omega^{\alpha}{}_{\nu}, \qquad \check{\omega}^{\alpha}{}_{\kappa}\, \check{\omega}^{\kappa}{}_{\mu}\, \check{\omega}^{\mu}{}_{\nu} = -\check{\omega}^{\alpha}{}_{\nu}$$

with the property of being related to a two-dimensional space, orthogonal to another two-dimensional space determined by the dual tensor $\check{\omega}_{\alpha\kappa}$. Then

$$F_{\mu\omega} = \sqrt{H}\left(\sin\frac{\theta}{2}\,\omega_{\mu\nu} + \cos\frac{\theta}{2}\,\check{\omega}_{\mu\nu}\right)$$

The γ^{μ} matrices are transformed by Lorentz transformations (Schweber, 1964)

$$\Lambda\,\gamma^{\mu}\,\Lambda^{-1} = \gamma^{\nu}\,L_{\nu}{}^{\mu} \tag{2}$$

where Λ is the spinorial Dirac representation of the Lorentz group and the $L^{\mu}{}_{\nu}$ are the components of the tensorial representation. We write the differential equation of the Lorentz group in the tensorial representation,

$$\frac{d}{ds} L^{\mu}{}_{\nu}(s) = F^{\mu}{}_{\alpha}\, L^{\alpha}{}_{\nu}(s), \qquad L^{\nu}{}_{\alpha}(0) = \delta^{\nu}_{\alpha}$$

where $F_{\mu\nu}$ is an antisymmetric, constant tensor. The solution to the Lorentz group equation is $L^{\nu}{}_{\mu}(s) = \exp(s\,F^{\nu}{}_{\mu})$ (Gelfand *et al.*, 1963).

With no loss of generality in the nonnull case, let $H = 1$ and we obtain (Bazański, 1965; Piña, 1967)

$$\begin{aligned}
L^{\nu}{}_{\mu}(s) = {} & \omega^{\nu}{}_{\kappa}\omega^{\kappa}{}_{\mu} \cosh\left(s\sin\frac{\theta}{2}\right) - \check{\omega}^{\nu}{}_{\kappa}\check{\omega}^{\kappa}{}_{\mu}\cos\left(s\cos\frac{\theta}{2}\right) \\
& + \omega^{\nu}{}_{\mu}\sinh\left(s\sin\frac{\theta}{2}\right) + \check{\omega}^{\mu}{}_{\nu}\sin\left(s\cos\frac{\theta}{2}\right) \\
\Lambda(s) = {} & \frac{1}{2}\omega_{\mu\nu}\,\gamma^{\mu}\gamma^{\nu}\cos\left(\frac{s}{2}\cos\frac{\theta}{2}\right)\sinh\left(\frac{s}{2}\sin\frac{\theta}{2}\right) \\
& + \frac{1}{2}\check{\omega}_{\mu\nu}\,\gamma^{\mu}\gamma^{\nu}\sin\left(\frac{s}{2}\cos\frac{\theta}{2}\right)\cosh\left(\frac{s}{2}\sin\frac{\theta}{2}\right) \\
& + I\cos\left(\frac{s}{2}\cos\frac{\theta}{2}\right)\cosh\left(\frac{s}{2}\sin\frac{\theta}{2}\right) - \sigma\sin\left(\frac{s}{2}\cos\frac{\theta}{2}\right)\sinh\left(\frac{s}{2}\sin\frac{\theta}{2}\right)
\end{aligned} \tag{3}$$

For the null case,

$$L^{\nu}{}_{\mu}(s) = \delta^{\nu}{}_{\mu} + sF^{\nu}{}_{\mu} + \frac{s^2}{2} F^{\nu}{}_{\gamma}F^{\gamma}{}_{\mu}, \qquad \Lambda(s) = I + \frac{s}{4} F_{\alpha\kappa}\,\gamma^{\alpha}\gamma^{\kappa} \tag{4}$$

4. THE DIRAC EQUATION

Let the constants m, c, and $\hbar$ be the mass of the electron, the velocity of light, and the Planck constant divided by 2π. Moreover, let e be the charge of the electron and A_{μ} be the potential four-vector of the electromagnetic field. The minimally coupled Dirac equation of an electron interacting with an electromagnetic field is

$$\left(i\hbar \frac{\partial}{\partial x^{\mu}} - \frac{e}{c} A_{\mu}\right) \gamma^{\mu}\phi = mc\phi$$

This equation has four linear independent solutions, which are denoted with index j, ϕ_j, and are grouped in a nonsingular 4×4 matrix Φ having these spinors as columns,

$$\Phi = [\phi_1\phi_2\phi_3\phi_4], \qquad \left(i\hbar \frac{\partial}{\partial x^{\mu}} - \frac{e}{c} A_{\mu}\right)\gamma^{\mu}\, \Phi = mc\Phi$$

Any particular solution to the Dirac equation is given by the product

$$\phi = \Phi\psi, \qquad \text{where} \quad \psi \text{ is a constant spinor} \tag{5}$$

The four components of the spinor ψ are the coefficients of the linear combination of the four spinors ϕ_j, which is equal to the particular solution.

The noninteracting Dirac equation for $A = 0$ is transformed by taking the left product with the matrix Λ of the Lorentz transformation having the form (3) or (4). In the right-hand side (multiplied by mc) then appears the transformed spinor (Schweber, 1964), $\tilde{\phi} = \Lambda\phi$. The left-hand side, except for the factor $i\hbar$, becomes, by using equation (2), which transforms the Dirac matrices,

$$\tilde{x}^{\kappa} = L^{\kappa}{}_{\mu}x^{\mu}, \qquad \Lambda\gamma^{\mu}\,\Lambda^{-1} \frac{\partial}{\partial x^{\mu}}\,\Lambda\phi = \gamma^{\kappa}\, L_{\kappa}{}^{\mu} \frac{\partial}{\partial x^{\mu}}\,\Lambda\phi = \gamma^{\kappa} \frac{\partial}{\partial \tilde{x}^{\kappa}}\,\tilde{\phi}$$

This gives the transformed Dirac equation in the new coordinates with the transformed spinor, but with the same Dirac matrices,

$$i\hbar\,\gamma^{\mu} \frac{\partial}{\partial \tilde{x}^{\mu}}\,\tilde{\phi} = mc\tilde{\phi}$$

The Φ matrix, being formed by spinors, should be transformed in a different way, $\tilde{\Phi} = \Lambda \; \Phi \; \Lambda^{-1}$. This fact, however, is not in contradiction with the Dirac equation; notice that equation (5) is transformed as

$$\tilde{\phi} = \Lambda\phi = \Lambda\Phi\psi = \Lambda\Phi\Lambda^{-1}\,\Lambda\psi = \tilde{\Phi}\tilde{\psi}$$

A 4×4 complex matrix Φ has 32 real quantities in it. The eight complex components of vectors B_ν and $\tilde{B}^\nu$ have only 16 real quantities. One is faced with a case similar to the Lagrange–Hamilton change of formalism in classical mechanics when the number of coordinates is doubled, but the second-order Lagrange equations are transformed into first-order Hamilton equations.

Expanding the matrix Φ as in (1), we get

$$\left(i\hbar\,\frac{\partial}{\partial x^\alpha} - \frac{e}{c}\,A_\alpha\right) B^\alpha = mcB$$

$$\left(i\hbar\,\frac{\partial}{\partial x^\kappa} - \frac{e}{c}\,A_\kappa\right) B + \epsilon_{\mu\nu\alpha\kappa}\, g^{\lambda\alpha} \left(i\hbar\,\frac{\partial}{\partial x^\lambda} - \frac{e}{c}\,A_\lambda\right) B^{\mu\nu} = mcB_\kappa$$

$$-\left(i\hbar\,\frac{\partial}{\partial x^\lambda} - \frac{e}{c}\,A_\lambda\right)(\epsilon^{\lambda\kappa\mu\nu}\, B_\kappa + g^{\lambda\mu}\,\tilde{B}^\nu - g^{\lambda\nu}\tilde{B}^\mu) = 2mcB^{\nu\mu}$$

$$\left(i\hbar\,\frac{\partial}{\partial x^\nu} - \frac{e}{c}\,A_\nu\right)\left(2B^{\alpha\nu} + \tilde{B}g^{\alpha\nu}\right) = mc\tilde{B}^\alpha$$

$$\left(i\hbar\,\frac{\partial}{\partial x^\mu} - \frac{e}{c}\,A_\mu\right)\tilde{B}^\mu = mc\tilde{B}$$

Eliminating B, $B^{\mu\nu}$, and $\tilde{B}$, we get

$$\diamondsuit = -g^{\lambda\sigma}\left(\frac{\partial}{\partial x^\lambda} + i\,\frac{e}{c\hbar}\,A_\lambda\right)\left(\frac{\partial}{\partial x^\sigma} + i\,\frac{e}{c\hbar}\,A_\sigma\right), \qquad F_{\alpha\kappa} = \frac{\partial A_\kappa}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\kappa}$$

$$\frac{m^2c^2}{\hbar^2}\,B_\kappa = \diamondsuit B_\kappa - i\,\frac{e}{c\hbar}\,F_{\kappa\alpha}\,B^\alpha - i\,\frac{e}{c2\hbar}\,g_{\kappa\lambda}\,\epsilon^{\lambda\mu\nu\sigma}\,F_{\nu\sigma}B_\mu$$

$$\frac{m^2c^2}{\hbar^2}\,\tilde{B}_\kappa = \diamondsuit\tilde{B}_\kappa - i\,\frac{e}{c\hbar}\,F_{\kappa\alpha}\tilde{B}^\alpha + i\,\frac{e}{c2\hbar}\,g_{\kappa\lambda}\,\epsilon^{\lambda\mu\nu\sigma}F_{\nu\sigma}B_\mu$$

In particular, for the noninteracting case $A = 0$, one obtains the Klein–Gordon equations in terms of the d'Alambert operator (Cercignani, 1967),

$$\square = -g^{\mu\nu}\,\frac{\partial^2}{\partial x^\mu \partial x^\nu}, \qquad \square B_\nu = \frac{m^2c^2}{\hbar^2}\,B_\nu, \qquad \square\tilde{B}^\nu = \frac{m^2c^2}{\hbar^2}\,\tilde{B}^\nu$$

The decoupling of the interacting equations is performed by introducing two new quantities

$$C^{\mu} = B^{\mu} + i\tilde{B}^{\mu}, \qquad \tilde{C}^{\mu} = B^{\mu} - i\tilde{B}^{\mu}$$

$$\frac{m^2c^2}{\hbar^2} C_{\mu} = \lozenge C_{\mu} - i\frac{e}{c\hbar}(F_{\mu\nu} - i\check{F}_{\alpha\nu})C^{\nu}$$

$$\frac{m^2c^2}{\hbar^2} \tilde{C}_{\mu} = \lozenge \tilde{C}_{\mu} - i\frac{e}{c\hbar}(F_{\mu\nu} + i\check{F}_{\mu\nu})\tilde{C}^{\nu}$$

These equations can be obtained from the Lagrangian density

$$\mathrm{L} = -\left(\frac{\partial \tilde{C}^{*}_{\alpha}}{\partial x^{\kappa}} - i\frac{e}{c\hbar} A_{\kappa}\tilde{C}^{*}_{\alpha}\right)\left(\frac{\partial C^{\alpha}}{\partial x_{\kappa}} + i\frac{e}{c\hbar} A^{\kappa}C^{\alpha}\right)$$
$$+ i\frac{e}{c\hbar}\tilde{C}^{*}_{\alpha}(F^{\alpha\kappa} - i\check{F}^{\alpha\kappa})C_{\kappa} + \frac{m^2c^2}{\hbar^2}\tilde{C}^{*}_{\alpha}C^{\alpha} + \text{c.c.}$$

where the asterisk denotes the complex conjugate and c.c. denotes the complex conjugate of the previous expression.

5. CONCLUSION

The translation between the tensorial and the spinorial notations of the Dirac equation has been reviewed and used to relate the vectorial and spinorial representations of the Lorentz transformations. The vectorial form of the noninteracting Dirac equation reduces to the vectorial Klein–Gordon equation. However, when a similar translation is applied to the interaction with an electromagnetic field, the Dirac equation leads, in the tensorial language, to a generalization of the vector Klein–Gordon equation where the interaction with the electromagnetic field does not agree with the form used commonly in the literature.

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